

# The analog of the Schauder inequality for closed surfaces in Euclidean spaces

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## Abstract

The analog of the Schauder inequality for closed surfaces in Euclidean spaces is obtained in this article.

## Introduction

Let  $E^n$  be  $n$ -dimensional ( $n > 1$ ) Euclidean space. Let  $D$  be the finite domain in  $E^n$ ,  $\partial D$  be the boundary of  $D$ ,  $\bar{D}$  be the closure of  $D$ . Let  $(x^1, \dots, x^n)$  be the Cartesian coordinates in  $E^n$ .

**Definition 1** . We say, that function  $f$  on  $D$  is of class  $C^{k,s}(\bar{D})$ ,  $k \geq 0$ ,  $s \in (0, 1)$ , if it has continuous partial derivatives up to  $k$ -th order inclusively and bounded value

$$|f|_{(D)k,s} = \sum_{|i|=0}^k \sup_{x \in \bar{D}} |\partial^{|i|} f(x)| + \sum_{|i|=k} \sup_{x_1, x_2 \in D} \frac{|\partial^{|i|} f(x_1) - \partial^{|i|} f(x_2)|}{|x_1 - x_2|^s}. \quad (1)$$

Partial derivatives of function  $f(x)$  are denoted by  $\partial^{|i|} f(x) \equiv \frac{\partial^{|i|} f(x)}{\partial^{i_1} x^1 \dots \partial^{i_n} x^n}$ , where  $|i| = i_1 + \dots + i_n$  is the order of derivative.  $|x| = (\sum_{i=1}^n (x^i)^2)^{1/2}$ , where  $(x^1, \dots, x^n)$  are the coordinates of point  $x \in E^n$ .

The value  $|f|_{(D)k,s}$  we call the norm of function  $f$  in the space  $C^{k,s}(\bar{D})$ . It is known (see [1]) that the space  $C^{k,s}(\bar{D})$  with norm denoted by formula (1) is complete normed space.

We define the cylinder  $C_{R,L}$  in  $E^n$  by the following formula:

$$C_{R,L} = \left\{ x : \sum_{i=1}^{n-1} (x^i)^2 < R^2, -2LR < x^n < 2LR \right\},$$

where  $L = \text{const} > 0$ ,  $R = \text{const} > 0$ , and let  $x = (0, 0, \dots, 0)$  be called its center.

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**Definition 2** . Domain  $D$  is called strictly Lipschitzian if for every point  $x_0 \in \partial D$  it can be introduced the coordinates

$$u^k = \sum_{l=1}^n c_l^k (x^l - x_0^l), \quad k = \overline{1, n},$$

where  $\|c_l^k\|$  is orthogonal matrix such that the intersection of  $\partial D$  and the cylinder  $\bar{C}_{R,L}$  corresponding to the coordinates  $\{u^k\}$ , is given by the equation

$$u^n = \omega(u^n), \quad u^n \equiv (u^1, \dots, u^{n-1}),$$

where  $\omega(u^n)$  is Lipschitzian function for  $|u^n| \leq R$  with Lipschitz constant bounded by  $L$ , and

$$\bar{D} \cap \bar{C}_{R,L} = \left\{ u : |u^n| \leq R, \quad -2LR \leq u^n \leq \omega(u^n) \right\}.$$

The numbers  $R$  and  $L$  are fixed for the domain  $D$ .

Arbitrary convex domain is strictly Lipschitzian (see [1]).

Let  $x_0 = (x_0^1, \dots, x_0^n)$  be a point of  $\partial D$ , where surface  $\partial D$  has tangent plane.

**Definition 3** . We call  $(u^1, \dots, u^n)$  the specific local coordinate system with origin at the point  $x_0$  if the coordinates  $\{u^k\}$  and  $\{x^k\}$  satisfy the following equations:  $u^k = \sum_{l=1}^n c_l^k (x^l - x_0^l)$ ,  $k = \overline{1, n}$ , and the axis  $u^n$  is directed to the normal of  $\partial D$  at the point  $x_0$ , that is outward for  $D$ .

**Definition 4** . Domain  $D$  is called the domain of class  $C^{l,s}$ ,  $l \geq 1$ , if it is strictly Lipschitzian and the coordinates  $\{u^k\}$  that are given in the definition 2 are the specific local coordinates, the function  $u^n = \omega(u^n)$ , defining the equation of the surface  $\partial D$  is of class  $C^{l,s}(|u^n| \leq R)$ .

**Definition 5** . We will say that the boundary  $\partial D$  of domain  $D$  is of class  $C^{l,s}$  if for every point  $x_0 \in \partial D$  there can be introduced the the specific local coordinates such that the function  $u^n = \omega(u^n)$  is of class  $C^{l,s}(|u^n| \leq R)$ .

Let the boundary  $\partial D$  of the domain  $D$  is of class  $C^{l_1, s_1}$ ,  $s_1 \in (0; 1)$ . Let on  $\partial D$  be given the function  $\varphi(x)$ ,  $x \in \partial D$ .

**Definition 6** . We will say that function  $\varphi(x)$  is of class  $C^{l,s}(\partial D)$  if it as a function of the specific local coordinates  $u^n = (u^1, \dots, u^{n-1})$  introduced for every point  $x_0 \in \partial D$  is of class  $C^{l,s}(|u^n| \leq R)$ , where  $|u^n| \leq R$  is the base of cylinder corresponding to the point  $x_0$ .

**Definition 7** . Norm  $|\varphi|_{(\partial D)l,s}$  of function  $\varphi$ , given on the surface  $\partial D$  is called the greatest of the norms  $|\varphi(u^n)|_{(|u^n| \leq R)l,s}$ , calculated for all points  $x_0 \in \partial D$ .

Let  $F$  be the closed orientable hypersurface in Euclidean space  $E^{n+1}$ . Let  $(y^1, \dots, y^{n+1})$  be the Cartesian coordinates in  $E^{n+1}$ . Let  $U$  be arbitrary open set on  $F$ .

**Definition 8** . Couple  $(U, h)$  is called the admissible map of class  $C^{k,s}$  if:

- 1)  $h$  is the homeomorphism  $U$  on the open ball  $K_r$  of radius  $r > 0$  in  $E^n$ ;
- 2) the inverse mapping  $\bar{h}^{-1}(x) \equiv (f^1(x), \dots, f^{n+1}(x))$ ,  $x \in K_r$ , satisfies the condition:  $f^\alpha \in C^{k,s}(\bar{K}_r)$ ,  $\alpha = \overline{1, n+1}$ .

**Definition 9** .  $F$  is called the surface of class  $C^{k,s}$  if the following conditions hold:

- 1)  $F$  is the surface of class  $C^k$ ;
- 2) on  $F$ , there exists the finite aggregate  $\{(U_i, h_i)\}_{i=\overline{1, N}}$  of admissible maps of class  $C^{k,s}$ , where the collection of sets  $(U_i)_{i=\overline{1, N}}$  is open covering of  $F$ ;
- 3) if  $U_i \cap U_j \neq \emptyset$  then the mapping  $h_j \circ h_i^{-1}$  is diffeomorphism of class  $C^k$  of  $h_i(U_i \cap U_j)$  on set  $h_j(U_i \cap U_j)$ .

**Definition 10** . The aggregate  $\{(U_i, h_i)\}_{i=\overline{1, N}}$  of the admissible maps considered in definition 9 is called the admissible atlas of class  $C^{k,s}$  of hypersurface  $F$ .

**Definition 11** . Function  $f$  determined on surface  $F$ , is called the function of class  $C^{p,s}(F)$ ,  $p < k$ , if

- 1) on hypersurface  $F$ , there exists the admissible atlas  $\{(U_i, h_i)\}_{i=\overline{1, N}}$  of class  $C^{k,s}$  and
- 2)  $f \circ h_i^{-1} \in C^{p,s}(\bar{K}_{r_i})$ ,  $i = \overline{1, N}$ .

We fix the admissible atlas  $\{(U_i, h_i)\}_{i=\overline{1, N}}$  of class  $C^{k,s}$  of hypersurface  $F$ .

Let be given the function  $f \in C^{p,s}(F)$ . The norm of function  $f$  in space  $C^{p,s}(F)$  is determined by the following formula:

$$|f|_{(F)p,s} = \max_i |f \circ h_i^{-1}|_{(K_{r_i})p,s}.$$

We will prove that the obtained normed space is complete. Let  $\{f_m\}_{m=1}^\infty$  be the Cauchy sequence of functions  $f_m$  of class  $C^{p,s}(F)$ , therefore  $\forall \varepsilon > 0$  and for every natural number  $l$

$$|f_{m+l} - f_m|_{(F)p,s} < \varepsilon,$$

for all sufficiently large  $m$ . Then, from the definition of norm, we obtain

$$|f_{m+l} \circ h_i^{-1} - f_m \circ h_i^{-1}|_{(K_{r_i})p,s} < \varepsilon.$$

Since the function space  $C^{p,s}(\bar{K}_{r_i})$  is complete normed space, then the sequence of functions  $\{f_m \circ h_i^{-1}\}$  on  $\bar{K}_{r_i}$  has limit:  $f \circ h_i^{-1} = \lim_{m \rightarrow \infty} f_m \circ h_i^{-1}$ . Since  $f \circ h_i^{-1} \in C^{p,s}(\bar{K}_{r_i})$ ,  $\forall i = \overline{1, N}$ , then  $f \in C^{p,s}(F)$ .

## 1 Statement of the result.

Let  $F \in C^{3,s}$ , where  $s \in (0; 1)$ . Let  $\{(U_i, h_i)\}_{i=\overline{1, N}}$  be the admissible atlas  $F$  of class  $C^{3,s}$ . Let  $(U, h)$  be an arbitrary map from the collection  $\{(U_i, h_i)\}_{i=\overline{1, N}}$ . Then the hypersurface  $F$  on map  $(U, h)$  is determined by the following equation system:

$$y^\alpha \equiv h^{-1\alpha}(x) = f^\alpha(x^1, \dots, x^n), \alpha = \overline{1, n+1}, (x^1, \dots, x^n) \in K_r \quad (2)$$

Consider the differential operator  $A$  on  $F$  that, on map  $(U, h)$ , is defined by:

$$A = \sum_{k=1}^n \sum_{p=1}^n a^{kp} \partial_k (\partial_p) + \sum_{j=1}^n b^j \partial_j + c. \quad (3)$$

Let  $a^{kp} = a^{pk}$ , and the operator  $A$  is strictly elliptic on  $F$ , i. e.

$$\sum_{k=1}^n \sum_{p=1}^n a^{kp}(x) \zeta_k \zeta_p \geq \nu \sum_{k=1}^n (\zeta_k)^2, \quad \nu = \text{const} > 0, \quad \forall \zeta_k, \quad k = \overline{1, n}.$$

Let, on  $F$ , be given a function  $f$  of class  $C^{2,s}(F)$ . Then we have:

$$A(f \circ h^{-1}(x)) = \sum_{k=1}^n \sum_{p=1}^n a^{kp}(x) \partial_k (\partial_p (f \circ h^{-1}(x))) + \sum_{j=1}^n b^j(x) \partial_j (f \circ h^{-1}(x)) + c(x) f \circ h^{-1}(x),$$

where  $x \in K_r$  for every admissible map  $(U, h)$  of class  $C^{3,s}$ .

**Theorem 1** . Let function  $f$  be a solution of class  $C^{2,s}(F)$  of the problem:  $Af = \gamma$ , where  $c(x) \neq 0$  on  $F$ ,  $\gamma \in C^{0,s}(F)$ ,  $a^{kp} \in C^{0,s}(F)$ ,  $b^j \in C^{0,s}(F)$ ,  $c \in C^{0,s}(F)$ . Then the following inequality holds:

$$|f|_{(F)2,s} \leq M |\gamma|_{(F)0,s},$$

where the constant  $M$  depends on  $s, n$ , the surface  $F$ , the coefficients  $a^{kp}, b^j, c$  ( $k, p, j = \overline{1, n}$ ) and the admissible atlas on  $F$   $\{(U_i, h_i)\}_{i=\overline{1, N}}$  of class  $C^{3,s}$ .

## 2 Auxiliary conjectures.

**Note 1** : Let  $f$  be the function of class  $C^{2,s}(F)$ ,  $\{U_i, h_i\}$  be the admissible map on  $F$ ,  $h_i(U_i) = K_{r_i}$ . Let  $x_0 \in \partial K_{r_i}$ . Consider the specific local coordinates  $(x^1, \dots, x^n)$  for the point  $x_0$  where  $x^n = \omega(x^1, \dots, x^{n-1})$ . Then the intersection of the cylinder  $\bar{C}_{R,L}$  at the point  $x_0$  and the surface  $\partial K_{r_i}$  is given by:  $x^n = \omega(x^1, \dots, x^{n-1})$ ,  $|(x^1, \dots, x^{n-1})| \leq R$ . We assume that the specific local coordinates were introduced in the ball  $K_{r_i}$ .

Let  $O_i(x_0)$  be an open domain in  $K_{r_i}$  such that  $\bar{O}_i(x_0) \supset (\partial K_{r_i} \cap \bar{C}_{R,L})$ . Let  $B_{\rho_j}(x_0)$  be an open ball in  $E^n$  of radius  $\rho_j$  with center at the point  $x_0$ .

We will prove several lemmas before the theorem 1.

**Lemma 1** . There exist numbers  $R, L, Q$  and set collection  $\{O_i(x_0)\}_{i=\overline{1, N}}$  such that for every point  $x_0 \in \partial K_{r_i}$  and for all  $i = \overline{1, N}$  the following conditions hold:

- 1)  $\exists j \neq i$  such that  $h_i^{-1}(O_i(x_0)) \subset U_j$ .
- 2)  $\exists \rho_j > 0$  such that  $h_j(h_i^{-1}(O_i(x_0))) \subset B_{\rho_j}(x_0) \subset K_{r_j}$ , where  $\text{dist}(B_{\rho_j}(x_0), \partial K_{r_j}) \geq Q > 0$ .

Proof of lemma 1 follows from definition 10, compactness of  $\partial K_{r_j}$ , definition 2 and finiteness of covering  $\{U_i\}_{i=\overline{1,N}}$ , for sufficiently small numbers  $R$  and  $L$ .

We fix numbers  $R, L, Q$ , point  $x_0 \in \partial K_{r_i}$  and set collection  $\{O_i(x_0)\}_{i=\overline{1,N}}$ , that satisfy lemma 1.

**Lemma 2** . *Under the conditions of lemma 1, the following inequality holds:*

$$\sup_{|(x^1, \dots, x^{n-1})| \leq R} \left| f \circ h_i^{-1}(x^1, \dots, x^{n-1}, \omega(x^1, \dots, x^{n-1})) \right| \leq \sup_{u \in B_{\rho_j}} \left| f \circ h_j^{-1}(u) \right|.$$

**Proof.** Since there exists a diffeomorphism of class  $C^3$  of the neighborhood  $O_i(x_0)$  into the ball  $B_{\rho_j}(x_0)$ , then there exist mappings:  $u^p = k^p(x^1, \dots, x^n)$ ,  $p = \overline{1, n}$ ,  $\forall x \in O_i(x_0)$ ,  $x^p = g^p(u^1, \dots, u^n)$ ,  $\forall u \in h_j(h_i^{-1}(O_i(x_0)))$ . Hence for every point  $x \in O_i(x_0)$  there exists point  $u \in B_{\rho_j}(x_0)$  such that  $f \circ h_i^{-1}(x) = f \circ h_j^{-1}(u)$ . Lemma is proved.

**Lemma 3** . *The following inequality holds:*

$$\begin{aligned} \sup_{|(x^1, \dots, x^{n-1})| \leq R} \left| \frac{\partial}{\partial x^k} (f \circ h_i^{-1}(x^1, \dots, x^{n-1}, \omega(x^1, \dots, x^{n-1}))) \right| &\leq \\ &\leq M \max_{p=\overline{1,n}} \sup_{u \in B_{\rho_j}} \left| \frac{\partial}{\partial u^p} (f \circ h_j^{-1}(u)) \right|, \quad k = \overline{1, n-1}, \end{aligned}$$

where  $M = \text{const} > 0$ .

**Proof.** We have

$$\begin{aligned} \frac{\partial}{\partial x^k} (f \circ h_i^{-1}(x^1, \dots, x^{n-1}, \omega(x^1, \dots, x^{n-1}))) &= \\ = \frac{\partial}{\partial x^k} (f \circ h_j^{-1}(u^1, \dots, u^n)) &= \frac{\partial}{\partial u^p} (f \circ h_j^{-1}(u^1, \dots, u^n)) \frac{\partial k^p}{\partial x^k}, \end{aligned}$$

where the point  $(u^1, \dots, u^n) \in B_{\rho_j}(x_0)$ ,  $u^l = k^l(x^1, \dots, x^{n-1}, \omega(x^1, \dots, x^{n-1}))$ ,  $l = \overline{1, n}$ . Since the functions  $k^p \in C^3$  then the functions  $\frac{\partial k^p}{\partial x^k}$  are bounded on  $|(x^1, \dots, x^{n-1})| \leq R$ . Lemma 3 is proved.

**Lemma 4** . *The following inequality holds:*

$$\begin{aligned} \sup_{|(x^1, \dots, x^{n-1})| \leq R} \left| \frac{\partial^2}{\partial x^k \partial x^q} (f \circ h_i^{-1}(x^1, \dots, x^{n-1}, \omega(x^1, \dots, x^{n-1}))) \right| &\leq \\ \leq M_1 \left( \max_{p=\overline{1,n}} \sup_{u \in B_{\rho_j}} \left| \frac{\partial}{\partial u^p} (f \circ h_j^{-1}(u)) \right| + \max_{p,r=\overline{1,n}} \sup_{u \in B_{\rho_j}} \left| \frac{\partial^2}{\partial u^p \partial u^r} (f \circ h_j^{-1}(u)) \right| \right), \\ k, q = \overline{1, n-1}, \end{aligned}$$

where  $M_1 = \text{const} > 0$ .

**Proof.** We have

$$\begin{aligned} & \frac{\partial^2}{\partial x^k \partial x^q} \left( f \circ h_i^{-1}(x^1, \dots, x^{n-1}, \omega(x^1, \dots, x^{n-1})) \right) = \\ & = \frac{\partial^2}{\partial u^p \partial u^r} \left( f \circ h_j^{-1}(u^1, \dots, u^n) \right) \frac{\partial k^p}{\partial x^k} \frac{\partial k^r}{\partial x^q} + \frac{\partial}{\partial u^p} \left( f \circ h_j^{-1}(u^1, \dots, u^n) \right) \frac{\partial^2 k^p}{\partial x^k \partial x^q}, \end{aligned}$$

where the point  $(u^1, \dots, u^n) \in B_{\rho_j}(x_0)$ ,  $u^l = k^l(x^1, \dots, x^{n-1}, \omega(x^1, \dots, x^{n-1}))$ ,  $l = \overline{1, n}$ .

Since the functions  $k^p \in C^3$  therefore the functions  $\frac{\partial k^p}{\partial x^k}$  and  $\frac{\partial^2 k^p}{\partial x^k \partial x^q}$  are bounded on  $|(x^1, \dots, x^{n-1})| \leq R$ . Therefore we obtain the proof of lemma 4.

**Lemma 5** . *The following inequality holds:*

$$\begin{aligned} & \left| f \circ h_i^{-1}(x^1, \dots, x^{n-1}, \omega(x^1, \dots, x^{n-1})) \right|_{(|(x^1, \dots, x^{n-1})| \leq R)2,s} \leq \\ & \leq M_1 \left( |f \circ h_j^{-1}(u)|_{B_{\rho_j}2,s} + |f \circ h_j^{-1}(u)|_{B_{\rho_j}1,s} \right), \end{aligned}$$

where  $M_1 = \text{const} > 0$ .

Proof follows from lemmas 2, 3 and 4.

**Lemma 6** . *For any function  $f \in C^{2,s}(\bar{B}_{\rho_j})$  the following inequality holds:*

$$|f|_{(B_{\rho_j})1,s} \leq M_2 |f|_{(B_{\rho_j})2,s},$$

where  $M_2 = \text{const} > 0$ .

**Proof.** We have the inequality (see [1]):

$$\begin{aligned} & \sup_{u_1, u_2 \in B_{\rho_j}} \frac{|\partial(f(u_1)) - \partial(f(u_2))|}{|u_1 - u_2|^s} \leq \\ & \leq M_3 \left( \sum_{k=0}^2 \sup_{B_{\rho_j}} |\partial^k(f)| \right)^s \left( \sum_{k=0}^1 \sup_{B_{\rho_j}} |\partial^k(f)| \right)^{1-s} \leq M_3 \left( \sum_{k=0}^2 \sup_{B_{\rho_j}} |\partial^k(f)| \right), \end{aligned}$$

where  $M_3 = \text{const} > 0$ . Therefore we obtain the proof of lemma 6.

### 3 Proof of theorem 1.

By lemmas 5 and 6 we have the inequality:

$$\begin{aligned} & \left| f \circ h_i^{-1}(x^1, \dots, x^{n-1}, \omega(x^1, \dots, x^{n-1})) \right|_{(|(x^1, \dots, x^{n-1})| \leq R)2,s} \leq \\ & \leq M_4 |f \circ h_j^{-1}(u)|_{B_{\rho_j}2,s}, \end{aligned}$$

where  $M_4 = \text{const} > 0$ . Since the ball  $\bar{K}_{r_i}$  is compact, then for some  $j$  we obtain:

$$|f|_{\partial K_{r_i} 2, s} \leq M_5 |f|_{B_{\rho_j} 2, s} \quad (4),$$

where  $M_5 = \text{const} > 0$ .

Since function  $f$  is a solution of class  $C^{2,s}(F)$  of the problem  $Af = \gamma$  then we have the inequality (see [1]):

$$|f|_{B_{\rho_j} 2, s} \leq M(B_{\rho_j}) \left( |\gamma|_{K_{r_j} 0, s} + \max_{K_{r_j}} |f| \right),$$

where  $\bar{B}_{\rho_j} \subset K_{r_j}$ .

We have the inequality (see [1]):

$$\max_{K_{r_j}} |f| \leq \max \left( \max_{\partial K_{r_j}} |f|; \max_{K_{r_j}} \left| \frac{\gamma}{c} \right| \right).$$

Since  $F$  is compact then there exists point  $x_0 \in F$  such that  $|f(x_0)| = \max_F |f|$ . Therefore there exists neighborhood  $U_l$  such that  $x_0 \in U_l$ . Hence, we obtain:

$$\max_{K_{r_j}} |f| \leq \max_{K_{r_l}} |f| \leq \max_{K_{r_l}} \left| \frac{\gamma}{c} \right|. \quad (5)$$

We have the estimate (see [1]):

$$|f|_{K_{r_i} 2, s} \leq M_6 \left( |\gamma|_{K_{r_i} 0, s} + \max_{K_{r_i}} |f| + |f|_{\partial K_{r_i} 2, s} \right),$$

where  $M_6 = \text{const} > 0$ . Using inequalities (4) and (5), we finish the proof of theorem 1.

## References

1. O.A. Ladyjenskaya, N.N. Uraltseva. Linear and quasilinear equations of elliptic type. M: Nauka, 1973.